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1968 J. Phys. A: Gen. Phys. 1 280

(http://iopscience.iop.org/0022-3689/1/2/315)

J. PHYS. A (PROC. PHYS. SOC.), 1968, SER. 2, VOL. 1. PRINTED IN GREAT BRITAIN

Mixture of outgoing and incoming gravitational radiation: change of mass of the source of radiation†

M. A. ROTENBERG
The Negev Institute for Arid Zone Research, Beersheva, Israel
MS. received 6*th November* 1967

Abstract. It has been fairly well established theoretically in recent works that outward travelling spherical gravitational waves carry away energy from their source. Methods of successive approximations applied to the Einstein gravitational field equations of general relativity have been used for that purpose. The object of this paper is to show, by a double-series approximation method, a similar result for the general case of any mixture of outgoing and incoming radiation.

1. Introduction

It is now a fairly well-proved fact that, in the general theory of relativity, outgoing gravitational waves from an isolated cohesive material source have a real physical significance. In particular, by means of a double-series approximation method applied to the gravitational field equations‡

$$R_{ik} = 0 \tag{1.1}$$

for free space, the following result for sandwich waves was shown in works by Bonnor (1959), Rotenberg (1964), Bonnor and Rotenberg (1966).

In the second approximation to the field equations the source suffers a secular loss of gravitational mass, except for a very special type of oscillation of the source. This loss of mass is equal to the total energy flux of outgoing radiation as calculated from the linear approximation.

This was established for axi-symmetric sources, such as a system of two point masses made to oscillate by means of a light spring connecting them. Using any oscillating linear distribution as a model of the source, this paper sets out to show a similar result in the general case of any mixture of outgoing and incoming gravitational radiation. From the extended result it will be immediately deduced that stationary waves produce no total change of mass of the source, as expected.

The linear source is described in detail in § 2. In § 3 the double-parameter approximation method is presented, and the metric to be employed by this method for the linear source is given in § 4. In § 5 an appropriate external solution of the linear approximation is derived for the source. Section 6 is preliminary to the solution of the non-linear approximation in § 7, from which the main result, concerning variation in mass of the source, is deduced. For convenience, an appendix is set aside for the inclusion of the approximate field equations and their solution, corresponding to the metric chosen in § 4.

2. The source and receiver

We shall suppose that, in the linear approximation to (1.1), distance, time and mass retain their Newtonian meanings.

The source will be chosen as a linear cohesive distribution of matter of finite length vibrating along the axis Oz, of a (pseudo Galilean) rectangular Cartesian coordinate system Oxyz, with its centre of mass coinciding with the origin O. Then, if $\overset{s}{I}(t)$ is the sth moment

† This paper constituted partial contents of a thesis submitted by the author (1964) to the University of London for the degree of Ph.D.

‡ In this paper, unless otherwise stated or implied a Latin index runs from 1 to 4; a Greek index from 1 to 3. The summation convention applies to both indices.

at time t, of the source about the plane $z = 0$,

$$\overset{1}{I}(t) = 0. \tag{2.1}$$

We shall suppose that the source is made to execute arbitrary, smooth motion along Oz during the finite period $t_1 \leqslant t \leqslant t_2$. Then $\overset{s}{I}(t)$ is an arbitrary bounded function of t which satisfies (2.1) for $s = 1$ and which is (i) constant outside the interval $t_1 \leqslant t \leqslant t_2$ and (ii) single valued with unique derivatives of all orders in the interval $t_1 \leqslant t \leqslant t_2$.

In conclusion, the vibration of the system may be considered as partly the *cause* of the outgoing waves and partly the *effect* of the incoming waves, so that the system acts as a 'source and receiver' of the waves.

3. The double-parameter approximation method

We present here the double-series approximation method applicable to the external field of any isolated coherent material source.

Let m be the total mass of the source and a be any chosen constant having the dimensions of length (in units of customary physical dimensions), such as the time average radius of gyration of the source. Both m and a are defined in the Newtonian sense but in relativistic units assumed in this paper. Then the method of successive approximations is to involve the double-series expansion of the metric tensor

$$g_{ik} = \overset{(00)}{g}_{ik} + \sum_{p=1}^{\infty} \sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{g}_{ik} \tag{3.1}$$

in terms of the parameters m and a, $\overset{(ps)}{g}_{ik}$ (p, $s = 0, 1, 2, \ldots$) being independent of m and a (cf. Bonnor 1959, Rotenberg 1964, Bonnor and Rotenberg 1966); $\overset{(00)}{g}_{ik}$ is the value of g_{ik} for flat space–time. As will be shown in § 5, the solution of the linear approximation is, in fact, the part of (3.1) which is linear in m, namely the single-series expansion (5.14) in a.

Substituting the expansion (3.1) into the field equation (1.1) and equating to zero the coefficient of $m^p a^s$ we obtain a set of ten second-order differential equations to be called the (ps) *approximation*. The ten equations take the forms

$$\overset{(ps)}{\Phi}_{lm}(\overset{(ps)}{g}_{ik}) = \overset{(ps)}{\Psi}_{lm}(\overset{(qr)}{g}_{ik}) \qquad (q \leqslant p-1, r \leqslant s). \tag{3.2}$$

In these, the left-hand sides are linear in $\overset{(ps)}{g}_{ik}$ (and their derivatives), the right-hand sides are non-linear in $\overset{(qr)}{g}_{ik}$ ($q \leqslant p-1$, $r \leqslant s$) (and their derivatives) determined from earlier approximation steps. Thus any ($1s$) approximation is linear and homogeneous in $\overset{(1s)}{g}_{ik}$ and their derivatives ($\overset{(1s)}{\Psi}_{lm} = 0$) and, consequently, belongs to the linear approximation. (Of course, the linear approximation includes also the trivial (00) approximation corresponding to flat space–time.) For $p \geqslant 2$ the (ps) approximation is non-linear: the ($2s$) approximations ($s = 0, 1, 2, \ldots$) constitute the second approximation, the ($3s$) approximations ($s = 0, 1, 2, \ldots$) constitute the third approximation, and so forth.

As is already known, the 4-momentum of any bounded cohesive material source is conserved in the linear approximation (Rotenberg 1964, 1968†) and, therefore, in the ($1s$) approximations. A change in 4-momentum may occur in the second approximation, and it is our aim to show that there does, in general, take place in this approximation a secular variation in mass of the linear source chosen in § 2 (see § 7).

Finally, the solution of the (ps) approximation, the (ps) *solution*, is represented by $\overset{(ps)}{g}_{ik}$ which satisfy (3.2).

4. The metric

For solving the important approximations for the source chosen in § 2 (which is axially symmetric about Oz) we shall use the axi-symmetric metric

$$ds^2 = -A\,dr^2 - r^2(B\,d\theta^2 + \sin^2\theta C\,d\phi^2) + D\,dt^2 \tag{4.1}$$

† To be published.

where (r, θ, ϕ) are the (pseudo) spherical polar coordinates of the field-point P and A, B, C, D are functions of r, θ, t. This is the diagonalized form of the more general axi-symmetric metric

$$ds^2 = -A\,dr^2 - r^2(B\,d\theta^2 + \sin^2\theta C\,d\phi^2) + D\,dt^2 + 2rE\,dr\,d\theta + 2F\,dr\,dt + 2rG\,d\theta\,dt \tag{4.2}$$

where E, F, G are also functions of r, θ, t (see Bonnor 1959).

Expanding the coefficients of the diagonal metric (4.1) in the form (3.1) we have

$$\left.\begin{aligned} -g_{11} &= A &&= 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{A} \\ -g_{22} &= r^2 B &&= r^2\left(1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{B}\right) \\ -g_{33} &= r^2\sin^2\theta C &&= r^2\sin^2\theta\left(1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{C}\right) \\ g_{44} &= D &&= 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{D} \end{aligned}\right\} \tag{4.3}$$

$\overset{(ps)}{A}$, $\overset{(ps)}{B}$, $\overset{(ps)}{C}$, $\overset{(ps)}{D}$ being functions of r, θ, t. This notation will be used in the following sections.

5. The $(1s)$ approximations

We obtain here suitable solutions of the linear, $(1s)$, approximations corresponding to the diagonal metric (4.1). To do this we first derive, for any isolated coherent material distribution with its centre of mass taken as the origin O, an appropriate solution of the linear approximation to the field equations

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -8\pi T_{ik} \tag{5.1}$$

starting with (pseudo) Galilean coordinates $x_i = (x, y, z, t)$. We then apply the solution to the special source in § 2, transform to (pseudo) spherical polar coordinates and render diagonal the resulting approximate metric by an appropriate infinitesimal coordinate transformation $(r, \theta, \phi, t) \to (r^*, \theta^*, \phi^*, t^*)$. We proceed as follows.

For weak fields suppose that, in Galilean coordinates x_i,

$$g_{ik} = \eta_{ik} + \gamma_{ik}, \qquad \eta_{ik} = \eta^{ik} = \text{diag.}(-1, -1, -1, +1) \tag{5.2}$$

γ_{ik} being small. We introduce γ^*_{ik} by

$$\gamma^*_{ik} = \gamma_{ik} - \tfrac{1}{2}\eta_{ik}\eta^{ab}\gamma_{ab} \equiv \gamma_{ik} = \gamma^*_{ik} - \tfrac{1}{2}\eta_{ik}\eta^{ab}\gamma^*_{ab} \tag{5.3}$$

and select (pseudo) Galilean coordinates x_i satisfying the harmonic coordinate condition

$$\eta^{ab}\gamma^*_{ia,b} = 0 \tag{5.4}$$

where the comma denotes partial differentiation. The linearized form of the field equations (5.1) then reduces to a set of wave equations (Eddington 1924, § 57; Landau and Lifshitz 1962, § 101)

$$\eta^{ab}\gamma^*_{ik,ab} = -16\pi T_{ik}. \tag{5.5}$$

Their solution in Kirchhoff form for mixed, outgoing and incoming, radiation may be written as

$$\gamma^*_{ik} = \alpha\overset{(-)}{\gamma^*_{ik}} + \beta\overset{(+)}{\gamma^*_{ik}} \qquad (\alpha, \beta \geqslant 0, \alpha + \beta = 1) \tag{5.6}$$

where

$$\overset{(\mp)}{\gamma^*_{ik}} = -4\int_V r^{*-1}T_{ik}(\tilde{x}_\alpha, t \mp r^*)\,dv. \tag{5.7}$$

The notations in (5.6) and (5.7) are as follows: in (5.6) the quantities $\overset{(-)}{\gamma^*_{ik}}$ and $\overset{(+)}{\gamma^*_{ik}}$, representing respectively the retarded and advanced potential solutions of (5.5) and (5.4), correspond to the radiation emitted from, and absorbed by, the sources of the field; α, β are constants to be referred to as the *strengths* of the emitted and absorbed radiation, respectively. In (5.7) the integration is to be carried out over any fixed space volume V containing all the sources of the field, and r^* is the distance of the point $\tilde{\mathrm{P}}(\tilde{x}_\alpha)$ ($\tilde{x}_\alpha = \tilde{x}, \tilde{y}, \tilde{z}$), included by the space element $dv = d\tilde{x}\, d\tilde{y}\, d\tilde{z}$ of integration, from the field-point $\mathrm{P}(x_\alpha)$ ($x_\alpha = x, y, z$) of interest.

It will be convenient to have the *multipole expansion* for the solution (5.6) obtained by expanding the integrand in (5.7), by means of the Taylor theorem, so that r (the *fixed* distance OP) occurs instead of r^*. The expansion involves the moments of all orders of T_{ik} about the coordinate planes, which are defined for time t by

$$I_{ik/\lambda\mu\nu\ldots}(t) = \int_V \tilde{x}_\lambda x_\mu x_\nu \ldots T_{ik}(\tilde{x}_\alpha, t)\, dv \qquad (\eta^{ab}T_{ia,b} = 0).\dagger \tag{5.8}$$

In fact, if we employ the *dimensionless moments* $h_{ik/\lambda\mu\nu\ldots}(t)$ given by

$$\left.\begin{aligned} h_{\alpha\beta/\lambda_1\lambda_2\ldots\lambda_s} &= \frac{I_{\alpha\beta/\lambda_1\lambda_2\ldots\lambda_s}}{ma^{s+2}} \\ h_{\alpha 4/\lambda_1\lambda_2\ldots\lambda_s} &= \frac{I_{\alpha 4/\lambda_1\lambda_2\ldots\lambda_s}}{ma^{s+1}} \\ h_{44/\lambda_1\lambda_2\ldots\lambda_s} &= \frac{I_{44/\lambda_1\lambda_2\ldots\lambda_s}}{ma^{s}} \end{aligned}\right\} \tag{5.9}$$

which we suppose to be independent of the parameters m and a introduced from § 3, then the expansion turns out to be (Rotenberg 1964, 1968)

$$\gamma^*_{\alpha\beta} = -4ma^2r^{-1}\bar{h}_{\alpha\beta} - 4ma^3n_\lambda(r^{-1}\hat{h}'_{\alpha\beta/\lambda} + r^{-2}\bar{h}_{\alpha\beta/\lambda}) + m\mathrm{O}(a^4) \tag{5.10}$$

$$\begin{aligned} \gamma^*_{\alpha 4} = {} & -4ma^2n_\lambda(r^{-1}\hat{h}'_{\alpha 4/\lambda} + r^{-2}\bar{h}_{\alpha 4/\lambda}) \\ & -2ma^3\{n_\lambda n_\mu r^{-1}\bar{h}''_{\alpha 4/\lambda\mu} + (3n_\lambda n_\mu - \delta_{\lambda\mu})(r^{-2}\hat{h}'_{\alpha 4/\lambda\mu} + r^{-3}\bar{h}_{\alpha 4/\lambda\mu})\} + m\mathrm{O}(a^4) \end{aligned} \tag{5.11}$$

$$\begin{aligned} \gamma^*_{44} = {} & -4mr^{-1} \\ & -2ma^2\{n_\lambda n_\mu r^{-1}\bar{h}''_{44/\lambda\mu} + (3n_\lambda n_\mu - \delta_{\lambda\mu})(r^{-2}\hat{h}'_{44/\lambda\mu} + r^{-3}\bar{h}_{44/\lambda\mu})\} \\ & -\tfrac{2}{3}ma^3\{n_\lambda n_\mu n_\nu r^{-1}\hat{h}'''_{44/\lambda\mu\nu} + 3n_\lambda(2n_\mu n_\nu - \delta_{\mu\nu})r^{-2}\bar{h}''_{44/\lambda\mu\nu} \\ & + 3n_\lambda(5n_\mu n_\nu - 3\delta_{\mu\nu})(r^{-3}\hat{h}'_{44/\lambda\mu\nu} + r^{-4}\bar{h}_{44/\lambda\mu\nu})\} + m\mathrm{O}(a^4). \end{aligned} \tag{5.12}$$

Here $n_\lambda = x_\lambda/r = (x/r, y/r, z/r)$ and the notation

$$\bar{\psi}^{(n)} = \alpha\psi^{(n)}(t-r) + \beta\psi^{(n)}(t+r), \qquad \hat{\psi}^{(n)} = \alpha\psi^{(n)}(t-r) - \beta\psi^{(n)}(t+r) \tag{5.13}$$

where $^{(n)}$ (equivalent to n primes) denotes the nth derivative with respect to t ($n \geqslant 0$), has been applied to $h_{ik/\lambda\mu\nu\ldots}(t)$.

This solution, (5.10) to (5.12), which is the external solution (for the material distribution) of the linear approximation to (5.1) or (1.1), is referred to as the *multipole wave solution* (for the distribution) of the linear approximation. By virtue of (5.2) and (5.3) it may be written symbolically in the form

$$g_{ik} = \overset{(00)}{g_{ik}} + \sum_{s=0}^{\infty} ma^s \overset{(1s)}{g_{ik}}, \qquad \overset{(11)}{g_{ik}} = 0 \tag{5.14}$$

† The relation in brackets expresses the conservation law of 4-momentum in the linear approximation.

$\overset{(00)}{g}_{ik}, \overset{(1s)}{g}_{ik}$ being independent of m and a; $\overset{(00)}{g}_{ik}$ corresponds to flat space–time; $\overset{(11)}{g}_{ik}, \overset{(12)}{g}_{ik}, \overset{(13)}{g}_{ik}, \ldots, \overset{(1s)}{g}_{ik}, \ldots$ respectively constitute what are called the *dipole*, *quadrupole*, *octupole*, ..., 2^s-*pole*, ... wave solutions of the linear approximation (or the (11), (12), (13), ..., (1s), ... *waves*) appropriate to the sources; and the static part $\overset{(00)}{g}_{ik}+m\overset{(10)}{g}_{ik}$, which gives the linear approximation to the Schwarzschild solution, represents what is simply referred to as the *monopole* solution for the sources.

From the second of (5.14) it follows that an isolated oscillating system does not produce dipole waves as well as monopole waves. The lowest wave-like term in the multipole wave solution (5.14) is the one in ma^2, which corresponds to the quadrupole waves (cf. Boardman and Bergmann 1959, Bonnor 1963, Rotenberg 1964, 1968).

We now apply the multipole wave solution (5.10) to (5.12) to the particular system of § 2. It is easy to see that for this system the only non-vanishing components of T_{ik} are T_{33}, T_{34}, T_{44}. From (5.6) and (5.7) we therefore have $\gamma^*_{33}, \gamma^*_{34}, \gamma^*_{44}$ as the only non-zero γ^*_{ik}. These components of γ^*_{ik} can be expressed in terms of the ordinary moments $\overset{s}{I}$ (or 2^s-*pole* moments, $s = 0, 1, 2, \ldots$) of the source about the plane $z = 0$ (and their time derivatives). We shall actually express them in terms of the quantities $\overset{s}{h}$, the dimensionless 2^s-pole moments defined by

$$ma^s\overset{s}{h}(t) = \overset{s}{I}(t) = \int_{-\infty}^{\infty} z^s T_{44}(z, t)\, dz \qquad (s \geqslant 0) \tag{5.15}$$

and satisfying

$$\overset{s}{h}{}^{(n)}(t) = 0 \qquad (n \geqslant 1)(t \leqslant t_1, t \geqslant t_2). \tag{5.16}$$

This is done with the assistance of the following two relations, proved at the end of this section:

$$\overset{s}{h}_{33}(t) = \frac{1}{(s+1)(s+2)}\overset{s+2}{h}{}''(t), \qquad \overset{s}{h}_{34}(t) = -\frac{1}{s+1}\overset{s+1}{h}{}'(t) \qquad (s \geqslant 0) \tag{5.17}$$

where

$$ma^{s+2}\overset{s}{h}_{33}(t) = \int_{-\infty}^{\infty} z^s T_{33}(z, t)\, dz, \qquad ma^{s+1}\overset{s}{h}_{34}(t) = \int_{-\infty}^{\infty} z^s T_{34}(z, t)\, dz \quad (s \geqslant 0). \tag{5.18}$$

We proceed by putting $\alpha = \beta = \lambda = \mu = \nu = \ldots = 3$ in (5.10) to (5.12) and employing (5.17): the result, expressed explicitly up to terms in a^3, is

$$\gamma^*_{33} = -2ma^2r^{-1}\underset{2}{\bar{h}}'' - \tfrac{2}{3}ma^3 \cos\theta(r^{-1}\underset{3}{\hat{h}}''' + r^{-2}\underset{3}{\bar{h}}'') + m\mathrm{O}(a^4) \tag{5.19}$$

$$\begin{aligned}\gamma^*_{34} = {} & 2ma^2 \cos\theta(r^{-1}\underset{2}{\hat{h}}'' + r^{-2}\underset{2}{\bar{h}}') \\ & + \tfrac{2}{3}ma^3\{\cos^2\theta r^{-1}\underset{3}{\bar{h}}''' + (3\cos^2\theta - 1)(r^{-2}\underset{3}{\hat{h}}'' + r^{-3}\underset{3}{\bar{h}}')\} + m\mathrm{O}(a^4)\end{aligned} \tag{5.20}$$

$$\begin{aligned}\gamma^*_{44} = {} & -4mr^{-1} - 2ma^2\{\cos^2\theta r^{-1}\underset{2}{\bar{h}}'' + (3\cos^2\theta - 1)(r^{-2}\underset{2}{\hat{h}}' + r^{-3}\underset{2}{\bar{h}})\} \\ & - \tfrac{2}{3}ma^3\{\cos^3\theta r^{-1}\underset{3}{\hat{h}}''' + 3(2\cos^3\theta - \cos\theta)r^{-2}\underset{3}{\bar{h}}'' \\ & + 3(5\cos^3\theta - 3\cos\theta)(r^{-3}\underset{3}{\hat{h}}' + r^{-4}\underset{3}{\bar{h}})\} + m\mathrm{O}(a^4)\end{aligned} \tag{5.21}$$

where the notation (5.13) applies to $\underset{s}{h} \overset{\mathrm{def}}{=} \overset{s}{h}$.

This approximate metric corresponds to Galilean coordinates $x_i = (x, y, z, t)$. We now transform from these coordinates to coordinates (r, θ, ϕ, t) and reduce the resulting approximate metric to diagonal form by means of the coordinate transformation

$$\left.\begin{aligned} r &= r^* - m + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{\alpha}(r^*, \theta^*, t^*), \qquad \theta = \theta^* + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{\beta}(r^*, \theta^*, t^*) \\ \phi &= \phi^*, \qquad t = t^* + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{\delta}(r^*, \theta^*, t^*) \end{aligned}\right\} \tag{5.22}$$

where $\overset{(1s)}{\alpha}(r, \theta, t)$, ... $(s \geqslant 2)$ are given by

$$\left.\begin{aligned} \overset{(12)}{\alpha} &= (1-\tfrac{5}{2}s^2)r^{-1}\underset{2}{\hat{h}'} - 2s^2r^{-2}\underset{2}{\bar{h}} - (1-\tfrac{1}{2}s^2)\int_{\infty}^{r} r^{-3}\underset{2}{\bar{h}}\,dr \\ \overset{(12)}{\beta} &= sc(-2r^{-2}\underset{2}{\hat{h}'} - r^{-3}\underset{2}{\bar{h}} + r^{-1}\int_{\infty}^{r} r^{-3}\underset{2}{\bar{h}}\,dr) \\ \overset{(12)}{\delta} &= (1+\tfrac{1}{2}s^2)r^{-1}\underset{2}{\bar{h}'} + (1-\tfrac{1}{2}s^2)(r^{-2}\underset{2}{\hat{h}} + 3r\int_{\infty}^{r} r^{-4}\underset{2}{\hat{h}}\,dr) \end{aligned}\right\} \tag{5.23}$$

$$\left.\begin{aligned} \overset{(13)}{\alpha} &= -(\tfrac{1}{3}c - \tfrac{7}{9}c^3)r^{-1}\underset{3}{\bar{h}''} - (\tfrac{4}{3}c - \tfrac{20}{9}c^3)(r^{-2}\underset{3}{\hat{h}'} + r^{-3}\underset{3}{\bar{h}}) \\ \overset{(13)}{\beta} &= -\tfrac{2}{3}sc^2r^{-2}\underset{3}{\bar{h}''} - (\tfrac{4}{3}s - \tfrac{5}{3}s^3)(r^{-3}\underset{3}{\hat{h}'} + r^{-4}\underset{3}{\bar{h}}) \\ \overset{(13)}{\delta} &= (\tfrac{1}{3}c - \tfrac{1}{9}c^3)(r^{-1}\underset{3}{\hat{h}''} + r^{-2}\underset{3}{\bar{h}'}) \end{aligned}\right\} \tag{5.24}$$

etc. in which $s = \sin\theta$, $c = \cos\theta$. If we omit terms involving $m^p a^s$ $(p \geqslant 2,\ s \geqslant 0)$ and the asterisks the result is

$$\left.\begin{aligned} g_{11} &= -1 - 2mr^{-1} - \sum_{s=2}^{\infty} ma^s \overset{(1s)}{A}, \qquad g_{22} = -r^2\Big(1 + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{B}\Big) \\ g_{33} &= -r^2\sin^2\theta\Big(1 + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{C}\Big), \qquad g_{44} = 1 - 2mr^{-1} + \sum_{s=2}^{\infty} ma^s \overset{(1s)}{D} \end{aligned}\right\} \tag{5.25}$$

where the (00) and (10) solutions appear explicitly on the right as

$$\overset{(00)}{g_{11}} = -1,\ \overset{(00)}{g_{22}} = -r^2,\ \overset{(00)}{g_{33}} = -r^2\sin^2\theta,\ \overset{(00)}{g_{44}} = 1 \tag{5.26}$$

$$\overset{(10)}{g_{11}} = -2r^{-1},\ \overset{(10)}{g_{22}} = 0,\ \overset{(10)}{g_{33}} = 0,\ \overset{(10)}{g_{44}} = -2r^{-1} \tag{5.27}$$

where the (11) (dipole) solution is non-existent,

$$\overset{(11)}{g_{ik}} = 0 \tag{5.28}$$

and where the $(1s)$ solutions $(s \geqslant 2)$ are given by

$$\left.\begin{aligned}
\overset{(12)}{A} &= 2s^2(r^{-1}\underset{2}{\bar{h}}'' + 3r^{-2}\underset{2}{\hat{h}}' + 3r^{-3}\underset{2}{\bar{h}}) \\
\overset{(12)}{B} &= s^2\left(r^{-1}\underset{2}{\bar{h}}'' - 3r^{-3}\underset{2}{\bar{h}} - 3r^{-1}\int_{\infty}^{r} r^{-3}\underset{2}{\bar{h}}\,dr\right) \\
\overset{(12)}{C} &= -s^2\left(r^{-1}\underset{2}{\bar{h}}'' + 4r^{-2}\underset{2}{\hat{h}}' + 5r^{-3}\underset{2}{\bar{h}} + r^{-1}\int_{\infty}^{r} r^{-3}\underset{2}{\bar{h}}\,dr\right) \\
\overset{(12)}{D} &= 2s^2(r^{-1}\underset{2}{\bar{h}}'' + r^{-2}\underset{2}{\hat{h}}') - 2(4-3s^2)r^{-3}\underset{2}{\bar{h}} - 12(2-s^2)r\int_{\infty}^{r} r^{-5}\underset{2}{\bar{h}}\,dr
\end{aligned}\right\} \tag{5.29}$$

$$\left.\begin{aligned}
\overset{(13)}{A} &= (\tfrac{1}{3}c - \tfrac{5}{9}c^3)(r^{-1}\underset{3}{\hat{h}}''' + 6r^{-2}\underset{3}{\bar{h}}'' + 15r^{-3}\underset{3}{\hat{h}}' + 15r^{-4}\underset{3}{\bar{h}}) \\
\overset{(13)}{B} &= \tfrac{1}{3}s^2cr^{-1}\underset{3}{\hat{h}}''' + (\tfrac{4}{3}c - \tfrac{10}{9}c^3)r^{-2}\underset{3}{\bar{h}}'' + (\tfrac{5}{3}c - \tfrac{5}{9}c^3)(r^{-3}\underset{3}{\hat{h}}' + r^{-4}\underset{3}{\bar{h}}) \\
\overset{(13)}{C} &= -\tfrac{1}{3}s^2cr^{-1}\underset{3}{\hat{h}}''' - (2c - \tfrac{20}{9}c^3)r^{-2}\underset{3}{\bar{h}}'' - (5c - \tfrac{55}{9}c^3)(r^{-3}\underset{3}{\hat{h}}' + r^{-4}\underset{3}{\bar{h}}) \\
\overset{(13)}{D} &= (\tfrac{1}{3}c - \tfrac{5}{9}c^3)(r^{-1}\underset{3}{\hat{h}}''' + 4r^{-2}\underset{3}{\bar{h}}'' + 9r^{-3}\underset{3}{\hat{h}}' + 9r^{-4}\underset{3}{\bar{h}})
\end{aligned}\right\} \tag{5.30}$$

etc. We can verify by direct substitution that the above (10), (12), (13) solutions satisfy the respective $(1s)$ approximations given in the appendix as (A1) to (A7) $(P = Q = \dots = N = 0)$ for the diagonal metric (4.3).

In subsequent work we require, in addition to the (10), (12), (13) solutions, the *form* of the (14) solution as far as terms of order r^{-2}. In fact, expressions for the appropriate $(1s)$ solution up to order r^{-2} for *any* $s \geqslant 2$ are found to be of the following forms:

$$\left.\begin{aligned}
\overset{(1s)}{A}\ (s\text{ even}) &= r^{-1}\overset{s}{S}_A(\theta)\underset{s}{\bar{h}}^{(s)} + r^{-2}\overset{s}{T}_A(\theta)\underset{s}{\hat{h}}^{(s-1)}, \dots \\
\overset{(1s)}{A}\ (s\text{ odd}) &= r^{-1}\overset{s}{S}_A(\theta)\underset{s}{\hat{h}}^{(s)} + r^{-2}\overset{s}{T}_A(\theta)\underset{s}{\bar{h}}^{(s-1)}, \dots
\end{aligned}\right\}. \tag{5.31}$$

Finally, we establish (5.17). The conservation equations in the parenthesis in (5.8) reduce, for the linear source, to

$$T_{33,3} = T_{34,4}, \quad T_{43,3} = T_{44,4} \qquad (T_{ik} \equiv T_{ik}(z,t)). \tag{5.32}$$

Multiplying the second of (5.32) by z^{s+1} and integrating along Oz between the limits $z = \pm\infty$, we have

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty} z^{s+1}T_{44}\,dz &= \int_{-\infty}^{\infty} z^{s+1}T_{43,3}\,dz = \int_{-\infty}^{\infty} \{(z^{s+1}T_{43})_{,3} - (s+1)z^sT_{43}\}\,dz \\
&= [z^{s+1}T_{43}]_{z=-\infty}^{\infty} - (s+1)\int_{-\infty}^{\infty} z^sT_{43}\,dz.
\end{aligned} \tag{5.33}$$

Since the T_{ik} vanish outside the source it follows that the first term on the extreme right is zero, and so we obtain the second of (5.17). Similarly, by multiplying the first of (5.32) by z^{s+1} and integrating along Oz between the limits $z = \pm\infty$, we get

$$\overset{s}{h}_{33}(t) = -\frac{1}{s+1}\overset{s+1}{h'_{34}}(t); \tag{5.34}$$

inserting in this the relation obtainable from the second of (5.17) by replacing s by $s+1$ gives the first of (5.17).

6. The second approximation

As the 4-momentum of the source is conserved in the linear, $(1s)$, approximations, we are led to consider the second, $(2s)$, approximations in search for a secular change in mass of the source. Of the $(2s)$ approximations we shall study those for which $s \leqslant 4$, the most important of these being the (24) approximation, which is the lowest one revealing a permanent change in mass of the source.

The exact solution of a $(2s)$ approximation, except for $s = 0, 1$, is extremely difficult to obtain. Our aim is to find a *permanent* change in the mass of the source resulting from the finite period of its vibration. Those parts of the $\overset{(2s)}{g}_{ik}$ which are *transient*, i.e. which are each the same after the oscillation as before, are of no interest and may be ignored in the analysis. Terms of order higher than r^{-1} may be ignored too, since, from the Schwarzschild solution

$$ds^2 = -(1-2mr^{-1})^{-1}\,dr^2 - r^2(d\theta^2 + s^2\,d\phi^2) + (1-2mr^{-1})\,dt^2 \tag{6.1}$$

it is evident that terms in the metric representing variation in gravitational mass during the interval $t_1 \leqslant t \leqslant t_2$ are of order r^{-1}.

The equations constituting any (ps) approximation are (A1) to (A7), where $\overset{(ps)}{P}, \overset{(ps)}{Q}, \ldots, \overset{(ps)}{N}$ on the right consist entirely of interaction terms known from previous approximations. Their solution is given by (A8) to (A11), which contain six functions of integration, (A12). The key to this solution is the value of $\overset{(ps)}{A}$ which satisfies the inhomogeneous wave equation (A8).

For $p \geqslant 2$, the (ps) solution (A8) to (A11) is indeterminate to the extent of a *complementary solution* of the (ps) approximation, i.e. a solution of

$$\Phi_{lm}(\overset{(ps)}{g}_{ik}) = 0 \tag{6.2}$$

where the Φ_{lm} here stand for the left-hand sides of (A1) to (A7). However, we shall suppose that the functions representing the essential sources of the wave field have already been chosen for the 2^s-pole wave solutions of the linear, $(1s)$, approximations

$$\Phi_{lm}(\overset{(1s)}{g}_{ik}) = 0. \tag{6.3}$$

No fresh source functions are to be used other than those which are necessary to satisfy the inhomogeneous equations (A1) to (A7), and which are non-singular for $r > 0$ and lead to Galilean conditions at infinity. For this reason, in the $(2s)$ approximations ($s \leqslant 4$), which are going to be examined, all the functions (A12) of integration, save possibly $\overset{(2s)}{\chi}(r, t)$, will be ignored, except for the purpose of choosing suitable lower limits of various integrals we meet in solving the $(2s)$ approximations. Functions of integration resulting from solution of (A8) will be treated similarly.

In considering the $(2s)$ approximations step by step from $s = 0$ to 4, we shall be interested only in non-transient terms of order not higher than r^{-1} in their solutions. As these do not yield terms of order higher than r^{-3} on the left of (A1) to (A7), we therefore do not need to evaluate $\overset{(2s)}{P}, \overset{(2s)}{Q}, \ldots, \overset{(2s)}{N}$ on the right of these equations further than terms of order r^{-3}. However, we shall at first retain in the solutions terms up to order as high as r^{-3}, except those in $\overset{(2s)}{A}$ of order exceeding r^{-1} that are not needed to satisfy the key equation (A8) up to order r^{-3}. This is for the purpose of verifying that the solutions thus obtained actually satisfy (A1) to (A7) up to order r^{-3}, i.e. make all terms up to order r^{-3} cancel each other on substitution back into equations (A1) to (A7).† After this verification, terms significant to a secular change in the mass of the source will be picked out.

Turning to the particular values of s from 0 to 4 we remark first that the (20) approximation does not interest us. This is simply because the solution $\overset{(p0)}{g}_{ik}$ of any $(p0)$ approximation is clearly the pth approximation (m^p contribution) to the (static) Schwarzschild metric (6.1) of a central mass m, to which the linear source reduces when $a = 0$. The (21) approximation vanishes because, from (5.28), there are no $\overset{(11)}{g}_{ik}$ to produce with $\overset{(10)}{g}_{ik}$ non-linear $\overset{(21)}{\Psi}_{lm}$ in (3.2) ($p = 2$, $s = 1$); thus, according to our agreement not to introduce fresh source functions when unnecessary, the $\overset{(21)}{g}_{ik}$ must be put equal to zero.

As for the (22) approximation, by virtue of (5.28) the terms in $\overset{(22)}{\Psi}_{lm}$ on the right of (3.2) ($p = s = 2$) only come from the combination

$$\overset{(10)}{g}_{ik} \times \overset{(12)}{g}_{ik} \tag{6.4}$$

of the $\overset{(1r)}{g}_{ik}$ (and their derivatives)‡ and, up to order r^{-3} at least, are linear in $\overset{2}{h}{}^{(n)}(t \mp r)$ $(n > 0)$. In fact, up to order r^{-3},

$$\left.\begin{aligned} &\overset{(22)}{P} = 4s^2r^{-3}\underset{2}{\hat{h}}''', \ \overset{(22)}{Q} = -\overset{(22)}{R} = 2s^2(2r^{-2}\underset{2}{\bar{h}}^{\text{iv}} + 5r^{-3}\underset{2}{\hat{h}}'''), \ \overset{(22)}{S} = 12s^2r^{-3}\underset{2}{\hat{h}}''' \\ &\overset{(22)}{L} = 8sc(r^{-2}\underset{2}{\hat{h}}''' + 3r^{-3}\underset{2}{\bar{h}}''), \ \overset{(22)}{M} = -8s^2r^{-3}\underset{2}{\bar{h}}''', \ \overset{(22)}{N} = 8sc(r^{-2}\underset{2}{\bar{h}}''' + 3r^{-3}\underset{2}{\hat{h}}'' \end{aligned}\right\}. \tag{6.5}$$

From (5.16) it is reasonable to assume that this linearity in $\overset{2}{h}{}^{(n)}(t \mp r)$ implies that no permanent change of order not exceeding r^{-1} is to be found in the (22) approximation to the metric. In confirmation of this, the (22) solution turns out to be (Rotenberg 1964)

$$\left.\begin{aligned} \overset{(22)}{A} &= -4s^2r^{-1}\ln r\,\underset{2}{\hat{h}}''' + 4(2-3s^2)r^{-2}\ln r\,\underset{2}{\bar{h}}'' - \tfrac{4}{3}s^2r^{-1}\underset{2}{\hat{h}}''' \\ \overset{(22)}{B} &= -2s^2r^{-1}\ln r\,\underset{2}{\hat{h}}''' - 4(2-3s^2)r^{-3}\ln r\,\underset{2}{\hat{h}}' - \tfrac{2}{3}s^2r^{-1}\underset{2}{\hat{h}}''' + 4s^2r^{-3}\underset{2}{\hat{h}}' \\ \overset{(22)}{C} &= 2s^2r^{-1}\ln r\,\underset{2}{\hat{h}}''' + 8s^2r^{-2}\ln r\,\underset{2}{\bar{h}}'' - 4(2-s^2)r^{-3}\ln r\,\underset{2}{\hat{h}}' \\ &\quad + \tfrac{2}{3}s^2r^{-1}\underset{2}{\hat{h}}''' + \tfrac{8}{3}s^2r^{-2}\underset{2}{\bar{h}}'' + \tfrac{28}{3}s^2r^{-3}\underset{2}{\hat{h}}' \\ \overset{(22)}{D} &= -4s^2r^{-1}\ln r\,\underset{2}{\hat{h}}''' + 4(2-s^2)r^{-2}\ln r\,\underset{2}{\bar{h}}'' - 16r^{-3}\ln r\,\underset{2}{\hat{h}}' \\ &\quad - \tfrac{4}{3}s^2r^{-1}\underset{2}{\hat{h}}''' - \tfrac{16}{3}s^2r^{-2}\underset{2}{\bar{h}}'' + (\tfrac{64}{3} - 8s^2)r^{-3}\underset{2}{\hat{h}}' \end{aligned}\right\} \tag{6.6}$$

—up to order r^{-3} this solution satisfies (A1) to (A7) with $\overset{(22)}{P}, \overset{(22)}{Q}, \ldots, \overset{(22)}{N}$ on the right given as in (6.5), and it corresponds to the choice given by

$$\overset{(22)}{\chi}_1 + r^{-1}\overset{(22)}{\chi} = \tfrac{64}{3}r^{-3}\underset{2}{\hat{h}}''' \tag{6.7}$$

† Such solutions ensure that up to the (24) approximation the total flux of 'phoney' matter (obtained by means of the energy tensor T_{ik}) across a large sphere, centre the origin and radius r, is of order r^{-2}, and therefore zero across an infinite sphere. Thus, as far as secular variation of the mass of the source is concerned, the approximate solutions and the corresponding exact solutions of the $(2s)$ approximations ($s \leqslant 4$) should give the same result (see Bonnor 1959, § 11).

‡ The combination $\overset{(1m)}{g}_{ik} \times \overset{(1n)}{g}_{ik}$ of $\overset{(1r)}{g}_{ik}$ (and their derivatives), we say, represents the (2^m-*pole*) $\times$ (2^n-*pole*) *interaction* of the waves.

for the function $\overset{(22)}{\chi}(r, t)$ of integration in (A12), which is necessary to prevent singularities on the axis Oz of symmetry (except at O). The absence of any permanent change of order not exceeding r^{-1} from the (22) metric is clearly seen from (6.6). Similarly no such change appears in the (23) metric.

It remains now to study the (24) approximation. This is done in the next section, in which it is concluded that there does generally occur in this approximation a secular variation in mass of the source on account of the quadrupole $\times$quadrupole interaction of the waves.

7. The (24) approximation: change of mass of the source

The non-linear terms in the (24) approximation, i.e. those comprising $\overset{(24)}{P}, \overset{(24)}{Q}, \ldots, \overset{(24)}{N}$ on the right of (A1) to (A7), come from the combinations

$$\overset{(10)}{g}_{ik} \times \overset{(14)}{g}_{ik}, \qquad \overset{(11)}{g}_{ik} \times \overset{(13)}{g}_{ik}, \qquad \overset{(12)}{g}_{ik} \times \overset{(12)}{g}_{ik} \tag{7.1}$$

of the $\overset{(1r)}{g}_{ik}$ (and their derivatives). Of these combinations we need consider only the last, for the following reason: by virtue of (5.28) the second combination vanishes. From (5.27) and (5.31), the first combination gives leading terms up to order r^{-3} of forms similar to those of the leading terms of the combination (6.4) (the non-linear contribution (6.5) in the (22) approximation), and like the latter combination yields no non-transient terms of order not higher than r^{-1} in the metric. By means of (5.29) the precise expressions for $\overset{(24)}{P}, \overset{(24)}{Q}, \ldots, \overset{(24)}{N}$ consisting only of the effective non-linear terms can be found by a lengthy but straightforward calculation: up to r^{-3} terms they are

$$\left.\begin{aligned}
\overset{(24)}{P} &= s^4\{r^{-2}(2\underset{2}{\bar h}''\underset{2}{\bar h}^{\text{iv}} - 4\underset{2}{\bar h}'''^2 + 5\underset{2}{\hat h}'''^2) + 2r^{-3}(2\underset{2}{\hat h}'\underset{2}{\bar h}^{\text{iv}} + 17\underset{2}{\bar h}''\underset{2}{\hat h}''' - 12\underset{2}{\hat h}''\underset{2}{\bar h}''')\}\\
\overset{(24)}{Q} &= s^4\{-r^{-2}(\underset{2}{\bar h}'''^2 - \underset{2}{\hat h}'''^2) + 4r^{-3}(\underset{2}{\hat h}'\underset{2}{\bar h}^{\text{iv}} + \underset{2}{\bar h}''\underset{2}{\hat h}''')\}\\
\overset{(24)}{R} &= -s^4\{r^{-2}(\underset{2}{\bar h}'''^2 - \underset{2}{\hat h}'''^2) + 4r^{-3}(\underset{2}{\hat h}'\underset{2}{\bar h}^{\text{iv}} - \underset{2}{\bar h}''\underset{2}{\hat h}''' + 2\underset{2}{\hat h}''\underset{2}{\bar h}''')\}\\
\overset{(24)}{S} &= s^4\{r^{-2}(2\underset{2}{\bar h}''\underset{2}{\bar h}^{\text{iv}} + 5\underset{2}{\bar h}'''^2 - 4\underset{2}{\hat h}'''^2) + 4r^{-3}(\underset{2}{\hat h}'\underset{2}{\bar h}^{\text{iv}} - 3\underset{2}{\bar h}''\underset{2}{\hat h}''' + 5\underset{2}{\hat h}''\underset{2}{\bar h}''')\}\\
\overset{(24)}{L} &= -2s^3c\{9r^{-2}\underset{2}{\bar h}''\underset{2}{\hat h}''' + r^{-3}(16\underset{2}{\hat h}'\underset{2}{\hat h}''' + 29\underset{2}{\bar h}''^2)\}\\
\overset{(24)}{M} &= -s^4\{r^{-2}(2\underset{2}{\bar h}''\underset{2}{\hat h}^{\text{iv}} + \underset{2}{\bar h}'''\underset{2}{\hat h}''') + r^{-3}(4\underset{2}{\hat h}'\underset{2}{\hat h}^{\text{iv}} + 11\underset{2}{\bar h}''\underset{2}{\bar h}''' - 2\underset{2}{\hat h}''\underset{2}{\hat h}''')\}\\
\overset{(24)}{N} &= 2s^3c\{9r^{-2}\underset{2}{\bar h}''\underset{2}{\bar h}''' + 28r^{-3}(\underset{2}{\hat h}'\underset{2}{\bar h}''' + \underset{2}{\bar h}''\underset{2}{\hat h}'')\}
\end{aligned}\right\}. \tag{7.2}$$

Using (7.2) in the key equation (A8) and setting $\overset{(24)}{\eta}$ and $\overset{(24)}{\sigma}$ as zero we get

$$\Box \overset{(24)}{A} = s^4\{\tfrac{7}{2}r^{-2}(\underset{2}{\hat h}'''^2 - \underset{2}{\bar h}'''^2) + r^{-3}(6\underset{2}{\hat h}'\underset{2}{\bar h}^{\text{iv}} + 31\underset{2}{\bar h}''\underset{2}{\hat h}''' - 18\underset{2}{\hat h}''\underset{2}{\bar h}''' + \hat{\hat{Y}})\} - (\overset{(24)}{\chi}_1 + r^{-1}\overset{(24)}{\chi}) \tag{7.3}$$

where we have employed $\hat{\hat{Y}}$, the last of the four notations

$$\left.\begin{aligned}
\bar{\bar{X}} &= \alpha^2 X(t-r) + \beta^2 X(t+r), & \hat{\hat{X}} &= \alpha^2 X(t-r) - \beta^2 X(t+r)\\
\bar{\bar{Y}} &= \left(\alpha^2 \int_{-\infty}^{t-r} + \beta^2 \int_{\infty}^{t+r}\right) X(\xi)\, d\xi, & \hat{\hat{Y}} &= \left(\alpha^2 \int_{-\infty}^{t-r} - \beta^2 \int_{\infty}^{t+r}\right) X(\xi)\, d\xi
\end{aligned}\right\} \tag{7.4}$$

in which

$$X(\xi) = \overset{2}{h}'''^2(\xi). \tag{7.5}$$

A solution of (7.3), non-singular in θ and satisfying (7.3) up to order r^{-3}, can be obtained by a method similar to that adopted by Rotenberg (1967) for a solution of a similar inhomogeneous wave equation. The result is the first of (7.7); this solution corresponds to the choice of $\overset{(24)}{\chi}$ given by

$$\overset{(24)}{\chi}_1 + r^{-1}\overset{(24)}{\chi} = \tfrac{4}{15} r^{-3}\{6(\underset{2}{\bar h}'\underset{2}{\hat h}^{\mathrm{iv}} + \underset{2}{\hat h}'\underset{2}{\bar h}^{\mathrm{iv}}) + 13(\underset{2}{\bar h}''\underset{2}{\hat h}''' + \underset{2}{\hat h}''\underset{2}{\bar h}''') + 2\hat{\hat{Y}}\} \tag{7.6}$$

and made to achieve non-singularity in θ in the solution. With the help of (A9) to (A11) the remaining non-vanishing $\overset{(24)}{g}_{ik}$ can be calculated up to order r^{-3}. The whole (24) solution thus found is (using the notation (7.4))

$$\left.\begin{aligned}
\overset{(24)}{A} &= -(\tfrac{2}{15}s^2 + \tfrac{1}{20}s^4) r^{-1}\hat{\hat{Y}} + R_A \\
\overset{(24)}{B} &= -(\tfrac{1}{15}s^2 + \tfrac{1}{30}s^4) r^{-1}\hat{\hat{Y}} - (\tfrac{1}{5}s^2 - \tfrac{3}{4}s^4) r^{-1}\int_\infty^r r^{-1}\hat{\hat{Y}}\,dr + R_B \\
\overset{(24)}{C} &= (\tfrac{1}{15}s^2 + \tfrac{1}{30}s^4) r^{-1}\hat{\hat{Y}} - (\tfrac{1}{15}s^2 - \tfrac{3}{20}s^4) r^{-1}\int_\infty^r r^{-1}\hat{\hat{Y}}\,dr + R_C \\
\overset{(24)}{D} &= \tfrac{4}{15} r^{-1}\hat{\hat{Y}} + \tfrac{8}{15}\int_\infty^r r^{-1}\bar{\bar{X}}\,dr - (\tfrac{4}{15} - \tfrac{2}{15}s^2 - \tfrac{1}{20}s^4) r \int_\infty^r r^{-2}\bar{\bar{X}}\,dr + R_D
\end{aligned}\right\} \tag{7.7}$$

where

$$\begin{aligned}
R_A = {} & -(\tfrac{1}{15}s^2 + \tfrac{1}{40}s^4) r^{-1}(6\underset{2}{\bar h}'\underset{2}{\hat h}^{\mathrm{iv}} + 6\underset{2}{\hat h}'\underset{2}{\bar h}^{\mathrm{iv}} + 13\underset{2}{\bar h}''\underset{2}{\hat h}''' + 13\underset{2}{\hat h}''\underset{2}{\bar h}''') \\
& + \tfrac{7}{8}s^4 r^{-2}(\underset{2}{\bar h}''^2 - \underset{2}{\hat h}''^2) + s^4 r^{-3}(\tfrac{3}{4}\underset{2}{\bar h}\underset{2}{\hat h}''' - \tfrac{3}{4}\underset{2}{\hat h}\underset{2}{\bar h}''' - \tfrac{21}{4}\underset{2}{\bar h}'\underset{2}{\hat h}'' + \tfrac{21}{4}\underset{2}{\hat h}'\underset{2}{\bar h}'')
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
R_B = {} & -(\tfrac{1}{30}s^2 + \tfrac{1}{60}s^4) r^{-1}(6\underset{2}{\bar h}'\underset{2}{\hat h}^{\mathrm{iv}} + 6\underset{2}{\hat h}'\underset{2}{\bar h}^{\mathrm{iv}} + 13\underset{2}{\bar h}''\underset{2}{\hat h}''' + 13\underset{2}{\hat h}''\underset{2}{\bar h}''') \\
& + r^{-2}\{(\tfrac{3}{5}s^2 + \tfrac{1}{4}s^4)(\underset{2}{\bar h}'\underset{2}{\bar h}''' + \underset{2}{\hat h}'\underset{2}{\hat h}''') + (\tfrac{7}{20}s^2 + \tfrac{1}{16}s^4)\underset{2}{\bar h}''^2 + (\tfrac{7}{20}s^2 - \tfrac{7}{16}s^4)\underset{2}{\hat h}''^2\} \\
& + r^{-3}\{\tfrac{1}{2}s^4(\underset{2}{\bar h}\underset{2}{\hat h}''' - \underset{2}{\hat h}\underset{2}{\bar h}''') - (\tfrac{3}{5}s^2 + \tfrac{15}{4}s^4)\underset{2}{\bar h}'\underset{2}{\hat h}'' - (\tfrac{3}{5}s^2 + \tfrac{11}{4}s^4)\underset{2}{\hat h}'\underset{2}{\bar h}''\} \\
& - (\tfrac{1}{4}s^2 - \tfrac{15}{16}s^4) r^{-1}\int_\infty^r r^{-2}(\underset{2}{\bar h}''^2 + \underset{2}{\hat h}''^2)\,dr
\end{aligned} \tag{7.9}$$

$$\begin{aligned}
R_C = {} & (\tfrac{1}{30}s^2 + \tfrac{1}{60}s^4) r^{-1}(6\underset{2}{\bar h}'\underset{2}{\hat h}^{\mathrm{iv}} + 6\underset{2}{\hat h}'\underset{2}{\bar h}^{\mathrm{iv}} + 13\underset{2}{\bar h}''\underset{2}{\hat h}''' + 13\underset{2}{\hat h}''\underset{2}{\bar h}''') \\
& + r^{-2}\{(\tfrac{1}{5}s^2 + \tfrac{1}{20}s^4)(\underset{2}{\bar h}'\underset{2}{\bar h}''' + \underset{2}{\hat h}'\underset{2}{\hat h}''') + (\tfrac{7}{60}s^2 + \tfrac{89}{80}s^4)\underset{2}{\bar h}''^2 + (\tfrac{7}{60}s^2 + \tfrac{49}{80}s^4)\underset{2}{\hat h}''^2\} \\
& + r^{-3}\{\tfrac{1}{2}s^4(\underset{2}{\hat h}\underset{2}{\bar h}''' - \underset{2}{\bar h}\underset{2}{\hat h}''') - (\tfrac{1}{5}s^2 - \tfrac{69}{20}s^4)\underset{2}{\bar h}'\underset{2}{\hat h}'' - (\tfrac{1}{5}s^2 - \tfrac{129}{20}s^4)\underset{2}{\hat h}'\underset{2}{\bar h}''\} \\
& - (\tfrac{1}{12}s^2 - \tfrac{3}{16}s^4) r^{-1}\int_\infty^r r^{-2}(\underset{2}{\bar h}''^2 + \underset{2}{\hat h}''^2)\,dr
\end{aligned} \tag{7.10}$$

$$\begin{aligned}
R_D = {} & -r^{-1}\{(\tfrac{2}{5}s^2 + \tfrac{3}{20}s^4)(\underset{2}{\bar h}'\underset{2}{\hat h}^{\mathrm{iv}} + \underset{2}{\hat h}'\underset{2}{\bar h}^{\mathrm{iv}}) + (\tfrac{13}{15}s^2 + \tfrac{13}{40}s^4)(\underset{2}{\bar h}''\underset{2}{\hat h}''' + \underset{2}{\hat h}''\underset{2}{\bar h}''')\} \\
& + r^{-2}\{(\tfrac{4}{5}s^2 + \tfrac{3}{10}s^4)(\underset{2}{\bar h}'\underset{2}{\bar h}''' + \underset{2}{\hat h}'\underset{2}{\hat h}''') + (\tfrac{7}{15}s^2 + \tfrac{21}{20}s^4)\underset{2}{\bar h}''^2 + (\tfrac{7}{15}s^2 - \tfrac{7}{10}s^4)\underset{2}{\hat h}''^2\} \\
& + r^{-3}\{\tfrac{3}{4}s^4(\underset{2}{\bar h}\underset{2}{\hat h}''' - \underset{2}{\hat h}\underset{2}{\bar h}''') + (\tfrac{8}{5} - \tfrac{12}{5}s^2 - \tfrac{123}{20}s^4)\underset{2}{\bar h}'\underset{2}{\hat h}'' + (\tfrac{8}{5} - \tfrac{12}{5}s^2 - \tfrac{33}{20}s^4)\underset{2}{\hat h}'\underset{2}{\bar h}''\} \\
& - \tfrac{4}{3}\int_\infty^r r^{-3}(\underset{2}{\bar h}''^2 + \underset{2}{\hat h}''^2)\,dr + r\int_\infty^r r^{-4}\{(2 - s^2 - \tfrac{5}{8}s^4)\underset{2}{\bar h}''^2 + (2 - s^2 - \tfrac{17}{8}s^4)\underset{2}{\hat h}''^2\}\,dr.
\end{aligned} \tag{7.11}$$

As can be verified, this solution satisfies, up to order r^{-3}, the (24) approximation given by (A1) to (A7) with $\overset{(24)}{P}, \overset{(24)}{Q}, \ldots, \overset{(24)}{N}$ as in (7.2).

It is hard to interpret the (24) solution (7.7) physically, but the coordinate transformation

$$\left.\begin{aligned} r &= r^* - m^2a^4\left\{(\tfrac{2}{15} - \tfrac{1}{15}s^{*2} - \tfrac{1}{40}s^{*4})\int_\infty^{r^*} \eta^{-1}\hat{\hat{Y}}(\eta, t^*)\,d\eta \right.\\ &\quad \left. + (\tfrac{4}{45} - \tfrac{1}{9}s^{*2} - \tfrac{1}{36}s^{*4})r^*\int_\infty^{r^*} \eta^{-2}\hat{\hat{Y}}(\eta, t^*)\,d\eta\right\} \\ \theta &= \theta^* + m^2a^4\left\{(\tfrac{2}{15}s^*c^* + \tfrac{1}{10}s^{*3}c^*)r^{*-1}\int_\infty^{r^*} \eta^{-1}\hat{\hat{Y}}(\eta, t^*)\,d\eta \right.\\ &\quad \left. + (\tfrac{4}{45}s^*c^* + \tfrac{1}{90}s^{*3}c^*)\int_\infty^{r^*} \eta^{-2}\hat{\hat{Y}}(\eta, t^*)\,d\eta\right\} \\ \phi &= \phi^*, \qquad t = t^* - \tfrac{2}{15}m^2a^4\left\{\bar{\bar{Y}}(r^*, t^*) + r^*\int_\infty^{r^*} \eta^{-1}\hat{\hat{X}}(\eta, t^*)\,d\eta\right\} \end{aligned}\right\} \tag{7.12}$$

($s^* = \sin\theta^*$, $c^* = \cos\theta^*$) transforms the solution to one that can be physically interpreted, as we shall soon see. The transformed (24) solution, omitting the asterisks, is

$$\left.\begin{aligned} -\overset{(24)}{g}_{11} &= \overset{(24)}{A} = -\tfrac{4}{15}r^{-1}\hat{\hat{Y}} + (\tfrac{8}{45} - \tfrac{2}{9}s^2 - \tfrac{1}{18}s^4)\int_\infty^r \eta^{-1}\bar{\bar{X}}(\eta, t)\,d\eta - R_A \\ -r^{-2}\overset{(24)}{g}_{22} &= \overset{(24)}{B} = (\tfrac{1}{15}s^2 + \tfrac{1}{30}s^4)\int_\infty^r \eta^{-1}\bar{\bar{X}}(\eta, t)\,d\eta - R_B \\ -r^{-2}s^{-2}\overset{(24)}{g}_{33} &= \overset{(24)}{C} = -(\tfrac{1}{15}s^2 + \tfrac{1}{30}s^4)\int_\infty^r \eta^{-1}\bar{\bar{X}}(\eta, t)\,d\eta - R_C \\ \overset{(24)}{g}_{44} &= \overset{(24)}{D} = \tfrac{4}{15}r^{-1}\hat{\hat{Y}} + \tfrac{8}{15}\int_\infty^r \eta^{-1}\bar{\bar{X}}(\eta, t)\,d\eta \\ &\quad + (\tfrac{2}{15}s^2 + \tfrac{1}{20}s^4)r\int_\infty^r \eta^{-2}\bar{\bar{X}}(\eta, t)\,d\eta + R_D \\ r^{-1}\overset{(24)}{g}_{12} &= \overset{(24)}{E} = (\tfrac{2}{9}sc + \tfrac{1}{9}s^3c)\int_\infty^r \eta^{-1}\bar{\bar{X}}(\eta, t)\,d\eta \\ \overset{(24)}{g}_{14} &= \overset{(24)}{F} = -(\tfrac{1}{15}s^2 + \tfrac{1}{40}s^4)\int_\infty^r \eta^{-1}\hat{\hat{X}}(\eta, t)\,d\eta \\ &\quad + (\tfrac{4}{45} - \tfrac{1}{9}s^2 - \tfrac{1}{36}s^4)r\int_\infty^r \eta^{-2}\hat{\hat{X}}(\eta, t)\,d\eta \\ r^{-1}\overset{(24)}{g}_{24} &= \overset{(24)}{G} = -(\tfrac{2}{15}sc + \tfrac{1}{10}s^3c)\int_\infty^r \eta^{-1}\hat{\hat{X}}(\eta, t)\,d\eta \\ &\quad - (\tfrac{4}{45}sc + \tfrac{1}{90}s^3c)r\int_\infty^r \eta^{-2}\hat{\hat{X}}(\eta, t)\,d\eta; \end{aligned}\right\} \tag{7.13}$$

the earlier approximations are not affected. Now, from the definitions (7.8) to (7.11) of $R_A, \ldots, R_D$ we immediately observe that these contributions on the right of (7.13) consist of terms, everyone of which is either of order r^{-2} or higher for all t, or of order r^{-1} but tends to zero as $t \to \pm\infty$. Thus $R_A, \ldots, R_D$ do not yield any non-transient change of order not exceeding r^{-1} in the metric. Furthermore, it can easily be shown that the integral expressions on the right of (7.13) which involve

$$\int_{\infty}^{r} \eta^{-n}\bar{\bar{X}}(\eta,t)\,d\eta, \qquad \int_{\infty}^{r} \eta^{-n}\hat{\hat{X}}(\eta,t)\,d\eta \qquad (n=1,2) \tag{7.14}$$

also do not contribute to any such change in the metric (Bonnor 1959, Rotenberg 1964). So, on neglect of R_A, ..., R_D and the integral expressions, the (24) solution (7.13) becomes

$$\overset{(24)}{g}_{11} = \overset{(24)}{g}_{44} = \tfrac{4}{15}r^{-1}\hat{\hat{Y}} = \tfrac{4}{15}r^{-1}\left\{\alpha^2\int_{-\infty}^{t-r}\overset{2}{h}{}'''^2(\xi)\,d\xi - \beta^2\int_{\infty}^{t+r}\overset{2}{h}{}'''^2(\xi)\,d\xi\right\} \tag{7.15}$$

on account of the fourth of (7.4) and (7.5). Given $r>0$, we have

$$\overset{(24)}{g}_{11} = \overset{(24)}{g}_{44} = \begin{cases} \tfrac{4}{15}\beta^2 r^{-1}\displaystyle\int_{t_1}^{t_2}\overset{2}{h}{}'''^2(\xi)\,d\xi & \text{for} \quad t < t_1 - r \\ \tfrac{4}{15}\alpha^2 r^{-1}\displaystyle\int_{t_1}^{t_2}\overset{2}{h}{}'''^2(\xi)\,d\xi & \text{for} \quad t > t_2 + r. \end{cases} \tag{7.16}$$

This result corresponds to an approximate Schwarzschild metric, with terms in r^{-n} $(n \geqslant 2)$ ignored, for a central mass $\tilde{m}$ given by

$$\tilde{m} = \begin{cases} -\tfrac{2}{15}\beta^2 m^2 a^4\displaystyle\int_{t_1}^{t_2}\overset{2}{h}{}'''^2(\xi)\,d\xi & \text{for} \quad t < t_1 - r \\ -\tfrac{2}{15}\alpha^2 m^2 a^4\displaystyle\int_{t_1}^{t_2}\overset{2}{h}{}'''^2(\xi)\,d\xi & \text{for} \quad t > t_2 + r. \end{cases} \tag{7.17}$$

Hence the above solution (7.13) of the (24) approximation shows that there occurs in this approximation a permanent variation

$$\Delta m \overset{\text{def}}{=} -\tfrac{2}{15}(\alpha-\beta)m^2a^4\int_{t_1}^{t_2}\overset{2}{h}{}'''^2(\xi)\,d\xi \qquad (\alpha+\beta=1) \tag{7.18}$$

in the mass of the source on account of the quadrupole $\times$quadrupole interaction of the waves represented by the third combination of (7.1). This variation in mass is, up to the (24) approximation, precisely equal and opposite to the total energy removed by the waves as calculated by means of the pseudo-tensor (Rotenberg 1964, 1968).

In conclusion, the result follows from (7.18) that no secular change of mass occurs, either when the vibration of the source is of the very special type characterized by $\overset{2}{h}{}''' \equiv 0$ or, as expected, when the waves of the field are stationary ($\alpha = \beta = \tfrac{1}{2}$).

Acknowledgments

The author is deeply grateful to Professor W. B. Bonnor for helpful discussions leading to the subject of this paper.

Appendix. The approximate field equations and their solution

The (ps) approximation corresponding to the diagonal metric (4.1), which is formed by equating to zero the coefficients of m^pa^s obtained on substitution of the expansions (4.3) into (1.1), is written out below. To save printing, the labels (ps), which should have been placed above all the capital letters, have been omitted in this appendix, except where confusion might arise without them.

$$2R_{11} = 0: \quad -A_{44}+B_{11}+C_{11}+D_{11}+2r^{-1}(-A_1+B_1+C_1) + r^{-2}(A_{22}+A_2\cot\theta) = P \tag{A1}$$

$$2r^{-2}R_{22} = 0: \quad B_{11}-B_{44}+r^{-1}(-A_1+3B_1+C_1+D_1) + r^{-2}(A_{22}-2A-B_2\cot\theta+2B+C_{22}+2C_2\cot\theta+D_{22}) = Q \tag{A2}$$

$$2r^{-2}\operatorname{cosec}^2\theta R_{33} = 0: \quad C_{11}-C_{44}+r^{-1}(-A_1+B_1+3C_1+D_1)$$
$$+r^{-2}(A_2\cot\theta-2A-B_2\cot\theta+2B+C_{22}+2C_2\cot\theta+D_2\cot\theta) = R \qquad \text{(A3)}$$
$$2R_{44} = 0: \quad A_{44}+B_{44}+C_{44}-D_{11}-2r^{-1}D_1-r^{-2}(D_{22}+D_2\cot\theta) = S \qquad \text{(A4)}$$
$$2R_{12} = 0: \quad -B_1\cot\theta+C_{12}+C_1\cot\theta+D_{12}-r^{-1}(A_2+D_2) = L \qquad \text{(A5)}$$
$$2R_{14} = 0: \quad B_{14}+C_{14}+r^{-1}(-2A_4+B_4+C_4) = M \qquad \text{(A6)}$$
$$2R_{24} = 0: \quad A_{24}-B_4\cot\theta+C_{24}+C_4\cot\theta = N. \qquad \text{(A7)}$$

A subscript 1, 2 or 4 after A, B, C or D denotes differentiation with respect to r, θ or t—unless otherwise implied this notation is to apply to any non-tensorial symbol. The left-hand sides of the above equations comprise terms linear in the $\overset{(ps)}{g}_{ik}$ (and their derivatives). The functions P, Q, ..., N on the right consist of terms non-linear in the $\overset{(qr)}{g}_{ik}$ ($q \leqslant p-1$, $r \leqslant s$) (and their derivatives), known from solutions of the earlier approximations, and these functions are zero in the linear, $(1s)$, approximations.

The (ps) approximation given above has been integrated by Rosen and Shamir (1957), Bonnor (1959), and we merely write out its solution. It is

$$\Box A \overset{\text{def}}{=} (A_{11}+2r^{-1}A_1)+r^{-2}(A_{22}+A_2\cot\theta)-A_{44}$$
$$= P-\int(M_1+r^{-1}M)\,dt-\int\{(L_1+r^{-1}L)-\int(N_{11}+r^{-1}N_1)\,dt\}\,d\theta$$
$$-(\eta_1+r^{-1}\eta)+\int(\sigma_{11}+r^{-1}\sigma_1)\,d\theta-(\chi_1+r^{-1}\chi) \qquad \text{(A8)}$$

$$C = -A+\operatorname{cosec}^2\theta\int sc\left[2A+r^{-1}\int\left\{2A+r\left(\int M\,dt+\eta\right)\right\}dr+r^{-1}\tau\right]d\theta$$
$$+\operatorname{cosec}^2\theta\int s^2\left(\int N\,dt+\sigma\right)d\theta+\mu\operatorname{cosec}^2\theta \qquad \text{(A9)}$$

$$B = -C+r^{-1}\int\left\{2A+r\left(\int M\,dt+\eta\right)\right\}dr+r^{-1}\tau \qquad \text{(A10)}$$

$$D = A+r\int\left[2r^{-2}A+r^{-1}\left\{\int\left(L-\int N_1\,dt-\sigma_1\right)d\theta+\chi\right\}\right]dr+r\nu \qquad \text{(A11)}$$

in which $s = \sin\theta$, $c = \cos\theta$ and

$$\eta \equiv \eta(r,\theta), \quad \sigma \equiv \sigma(r,\theta), \quad \chi \equiv \chi(r,t), \quad \nu \equiv \nu(\theta,t), \quad \tau \equiv \tau(\theta,t), \quad \mu \equiv \mu(r,t) \qquad \text{(A12)}$$

are six functions of integration. The key to this solution is the solution for A of the inhomogeneous wave equation (A8).

The six arbitrary functions (A12) must be chosen to meet the following two requirements: (i) the (ps) metric shall be Galilean at infinity, (ii) it shall be non-singular on the axis of symmetry Oz, except at O. A sufficient condition for the second requirement to be satisfied is that

$$\overset{(ps)}{A}, \quad \{\overset{(ps)}{B}+\overset{(ps)}{H}(r,t)+\overset{(ps)}{K}(r,t)\cos\theta\}\operatorname{cosec}^2\theta, \quad \{\overset{(ps)}{C}+\overset{(ps)}{H}(r,t)+\overset{(ps)}{K}(r,t)\cos\theta\}\operatorname{cosec}^2\theta, \quad \overset{(ps)}{D}$$
$$\text{be of class } C^2 \text{ near } \sin\theta = 0 \qquad \text{(A13)}$$

for some $\overset{(ps)}{H}$ and $\overset{(ps)}{K}$, non-singular for all $r > 0$ and all t. It is necessary, whenever the above (ps) solution is used, to substitute it back into the (ps) field equations to determine whether any additional conditions are to be imposed on the six arbitrary functions.

Now suppose (A13) is satisfied. Subject the metric (4.3) to the sequence of coordinate transformations

$$r = r^* + \tfrac{1}{2}m^p a^s r^* \overset{(ps)}{H}(r^*, t^*), \quad \theta = \theta^* + \tfrac{1}{2}m^p a^s \overset{(ps)}{K}(r^*, t^*) \sin\theta^*, \quad \phi = \phi^*, \quad t = t^* \tag{A14}$$

($p = 1, 2, \ldots; s = 0, 1, 2, \ldots$). Then, on omitting the asterisks, we obtain for the (ps) approximation to the non-diagonal metric (4.2) the coefficients $\overset{(ps)}{A}, \overset{(ps)}{B}, \ldots, \overset{(ps)}{G}$ satisfying the condition that

$$\overset{(ps)}{A}, \quad \overset{(ps)}{B}\operatorname{cosec}^2\theta, \quad \overset{(ps)}{C}\operatorname{cosec}^2\theta, \quad \overset{(ps)}{D}, \quad \overset{(ps)}{E}\operatorname{cosec}\theta, \quad \overset{(ps)}{F}, \quad \overset{(ps)}{G}\operatorname{cosec}\theta$$
$$\text{be of class } C^2 \text{ near } \sin\theta = 0. \tag{A15}$$

This is the more usual form of statement of the sufficient condition of regularity along Oz for the (ps) metric.

Every approximate solution obtained in this paper directly satisfies either (A13) or (A15).

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